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REDUCED COSSERAT CONTINUUM AS A POSSIBLE MODEL FOR GRANULAR MEDIUM

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Abstract. We considered a nonlinear reduced Cosserat continuum: an elastic medium, whose translations and rotations are independent, the force stress tensor is asymmetric and the couple stress tensor is zero. We suggested the reduced Cosserat continuum as a possible model for granular medium. Granular materials are ubiquitous in our daily lives. They play an important role in many industries, such as mining, agriculture, and construction. We considered a nonlinear reduced Cosserat continuum for reference and current configurations and obtained the set of equations for each configuration.

Keywords: reduced Cosserat continuum, current configuration, reference configuration, energy coupling tensors.

Introduction

It is usually assumed that the sizes of the solid particles are negligible compared to typical distances between the particles. By contrast, our concern is granular materials in which grain size and nearest-neighbour distance are roughly comparable. There is no “rotational spring” in granular materials keeping rotations of the neighbouring grains (in the simplest case, a solid is pictured as an array of point masses connected by springs). First scholars, who stated that the rotational and translational degrees of freedom of the grains must be treated on an equal footing, were L. M. Schwartz et al. (1984).

Usually, the account of the rotational degrees of freedom requires the introduction of the couple stresses. Such models are well-known: the moment theory of elasticity (Cosserat Continuum), moment theory of elasticity with constrained rotation (Cosserat Pseudocontinuum). There is a vast research on these models and we will not attempt to review it. Practical application of these models requires an experimental determination of a large number of additional constants in the constitutive equations. There are many models for granular media (Badanin 2012, Harris 2009, Heinrich et al. 1996, Kurbatskiy et al. 2011). In the recent papers (Grekova et al. 2004, Kulesh et al. 2009) the Reduced Cosserat Continuum was suggested as a possible model to describe granular materials. In this continuum translations and rotations are independent, the stress tensor is not symmetric, and the couple stresses are zero. Note feature of this medium, which in the static limit reduced Cosserat model, turns into the classical continuum. The research of this model began not so long ago (Harris 2006, Grekova 2012a).

Granular materials are ubiquitous in our daily lives. They play an important role in many industries, such as mining, agriculture, and construction. They clearly are also important for geological processes (Grekova 2012b). Such model systems are a useful staring point in the description of ocean sediments and sedimentary rocks. In many cases, the foundations of buildings and underground facilities are located in non-cohesive soils. Therefore, researches of the behaviour of these soils are of great scientific and practical interest.

In this paper, we further develop (Lalin et al. 2011, 2012; Zdanchuk et al. 2010) a reduced Cosserat continuum as a possible model for granular medium. Earlier we presented linear reduced Cosserat continuum equations, plane wave’s propagation for the isotropic case, dispersion curves and now we would like to present nonlinear reduced Cosserat continuum equations in the current and reference configuration.

In the reduced Cosserat continuum each particle has 6 degrees of freedom, in terms of kinematics its state is described by vector r and rotation tensor P. Rotation tensor is a orthogonal tensor, its determinant is equal to 1 and is defined by 3 independent parameters. So kinematic state of the medium is described by fields $\mathbf{r}(x^s,t)$ and $\mathbf{P}(x^s,t)$, where $x^s (s=1,2,3)$ coordinates of the medium in the reference configuration (RC), t – time. Usually RC is selected as a known position of the body at the initial time t = 0. Let $\mathbf{r}(x^s,0)=\mathbf{R}(x^s)$. We introduce the basis $\mathbf{R}_k(x^s)=\partial\mathbf{R}/\partial x^k$, the dual basis $\mathbf{R}^k(x^s)$ and the Hamiltonian in the RC $\overset{\circ}{\nabla}=\mathbf{R}^s\dfrac{\partial}{\partial x^s}$.

The current position of the body at time t is called the current configuration (CC). We introduce the basis $\mathbf{r}_k(x^s,t)=\partial\mathbf{r}/\partial x^k$, the dual basis $\mathbf{r}^k(x^s,t)$ and the Hamiltonian in the CC $\nabla=\mathbf{r}^k\dfrac{\partial}{\partial x^k}$. The basis in the RC is not time-dependent and in the CC it is.

The purpose of this work is to obtain the equations of the nonlinear reduced Cosserat continuum as Lagrangian description in the RC and Eulerian in the CC.

System of equations in the current configuration

To establish the system of equation for the CC it is necessary to use the following equations:
the linear momentum balance equation

$$\frac{d}{dt}\int_V \rho \mathbf{v} dV = \int_S \mathbf{n}\cdot\boldsymbol{\tau} dS, \tag{1}$$

the kinetic momentum balance equation

$$\frac{d}{dt}\int_V (\rho \mathbf{J}\cdot\boldsymbol{\omega} + \mathbf{r}\times\rho\mathbf{v}) dV = \int_S \mathbf{r}\times(\mathbf{n}\cdot\boldsymbol{\tau}) dS \tag{2}$$

the energy balance equation:

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho\mathbf{v}^2 + \frac{1}{2}\omega\cdot\rho\mathbf{J}\cdot\boldsymbol{\omega} + \rho\Pi\right) dV = \int_S \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{v} dS \tag{3}$$

and the Reynolds transport theorem

$$\frac{d}{dt}\int_V \rho A dV = \int_V \rho \dot{A} dV \tag{4}$$

where ρ – the density in CC, $\mathbf{v}$ – the velocity vector $(\mathbf{v} = \dot{\mathbf{r}})$, $\mathbf{r}$ – the radius vector, $\boldsymbol{\omega}$ – the angular velocity vector $(\dot{\mathbf{P}} = \boldsymbol{\omega}\times\mathbf{P})$, $\mathbf{n}$ – an outward unit normal to the surface S, $\mathbf{J}$ – the mass density of an inertia tensor, Π – the mass density of the strain energy, $\dot{(...)} = \frac{\partial}{\partial t}(...) + \mathbf{v}\cdot\nabla(...)$ -the material time derivative, A - an arbitrary scalar, vector or tensor field, V- a volume limited by a surface S. To simplify calculations we assume that the body forces are equal to zero.

We shall combine the equation (4), the Gauss-Ostrogradskii theorem and equations (1) and (2). As a result we get motion equations for the CC:

$$\nabla\cdot\boldsymbol{\tau} = \rho\dot{\mathbf{v}} \tag{5}$$

$$\boldsymbol{\tau}_x = \rho(\boldsymbol{\omega}\times\mathbf{J}\cdot\boldsymbol{\omega} + \mathbf{J}\cdot\dot{\boldsymbol{\omega}}), \tag{6}$$

where $\boldsymbol{\tau}_x$ denotes the vector invariant of the tensor $\boldsymbol{\tau}$.

The definition of the vector invariant was given by Lurie (Lurie 1990).

From equations (3) and (4) using the identity $\nabla\cdot(A\cdot a) = \nabla\cdot A\cdot a + A^T\cdot\cdot\nabla a$ we get

$$\int_V \rho(\dot{\mathbf{v}}\cdot\mathbf{v} + \dot{\boldsymbol{\omega}}\cdot\mathbf{J}\cdot\boldsymbol{\omega} + \dot{\Pi}) dV = \int_V (\nabla\cdot\boldsymbol{\tau}\cdot\mathbf{v} + \boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v}) dV \tag{7}$$

Because V is arbitrary, using the expression (7) and the motion equation (5) we obtain the following relation

$$\rho\dot{\Pi} = \boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v} - \rho(\mathbf{J}\cdot\dot{\boldsymbol{\omega}})\cdot\boldsymbol{\omega} \tag{8}$$

Then we transform the second summand in the equation (8) with the motion equation (7) and the expression $(\omega\times J\cdot\omega)\cdot\omega = 0$. As a result we get

$$-\rho(\mathbf{J}\cdot\dot{\boldsymbol{\omega}})\cdot\boldsymbol{\omega} = \boldsymbol{\tau}_x^T\cdot\boldsymbol{\omega} = \boldsymbol{\tau}^T\cdot\cdot(\mathbf{I}\times\boldsymbol{\omega})$$

This equality was obtained using the expression $(\mathbf{A}\cdot\mathbf{B})_x\cdot\mathbf{a} = \mathbf{A}\cdot\cdot(\mathbf{B}\times\mathbf{a})$ with **B = I** (Lalin 2007). Now we can write the equation (8) like

$$\rho\dot{\Pi} = \boldsymbol{\tau}^T\cdot\cdot(\nabla\mathbf{v} + \mathbf{I}\times\boldsymbol{\omega}). \tag{9}$$

The strain state for the reduced Cosserat continuum is described by the strain tensor $\mathbf{e}(x^k, t)$. We define it for the CC:

$$\mathbf{e} = \mathbf{I} - \mathbf{F}^{-T}\cdot\mathbf{P}^T, \tag{10}$$

where $\mathbf{F}$ should satisfy a relation $\mathbf{F}^{-1} = \nabla\mathbf{R}^T$.

Let's differentiate the expression (10) with respect to time, use $\dot{\mathbf{F}}^{-T} = -\nabla\mathbf{v}\cdot\mathbf{F}^{-T}$ and $\dot{\mathbf{P}}^T = -\mathbf{P}^T\times\boldsymbol{\omega}$:

$$\dot{\mathbf{e}} + \mathbf{e}\times\boldsymbol{\omega} + \nabla\mathbf{v}\cdot\mathbf{e} = \nabla\mathbf{v} + \mathbf{I}\times\boldsymbol{\omega} \tag{11}$$

We obtain the compatibility equation (11) for the CC, which holds identically if strains and velocities are expressed by **r** and **P**.

After having introduced strain and stress, it is necessary to establish the relation between them, which is done through constitutive equations. Substitute the equation (11) in the expression (9), which lead to the following relation:

$$\rho\dot{\Pi} = \boldsymbol{\tau}^T\cdot\cdot(\dot{\mathbf{e}} + \mathbf{e}\times\boldsymbol{\omega} + \nabla\mathbf{v}\cdot\mathbf{e}) \tag{12}$$

Lalin (2007) in his paper showed that the expression $\dot{\mathbf{B}} + \mathbf{B}\times\boldsymbol{\omega} + \nabla\mathbf{v}\cdot\mathbf{B}$ is an objective (co-rotational or frame indifferent) derivative of tensor $\mathbf{B}$. And that the expression (12) provides energy coupling tensors $\boldsymbol{\tau}$ and $\mathbf{e}$. Therefore, the constitutive equations for the CC:

$$\rho\frac{\partial\Pi}{\partial\mathbf{e}} = \boldsymbol{\tau}. \tag{13}$$

The system of equation for the CC will not be full without the mass conservation law (Zhilin 1996)

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0, \tag{14}$$

System of equation for the CC contains the following unknown functions: 9 stresses $\boldsymbol{\tau}$, 9 strains $\mathbf{e}$, 6 velocities $\mathbf{v}, \boldsymbol{\omega}$ and density ρ. As a result we have 25 unknown functions. Corresponding equations are: 6 motion equations (5), (6), 9 compatibility equations (11), 9 constitutive equations (13) and 1 mass conservation law (14). In total there are 25 equations. The problem becomes fully set after adding the boundary and initial conditions. Our statement of the problem for the CC does not include any unknown kinematic **r**, **P** as well as the strain gradient **F**. The unknown **r**, **P** can be found by integrating equations $\mathbf{v} = \dot{\mathbf{r}}$, $\dot{\mathbf{P}} = \boldsymbol{\omega}\times\mathbf{P}$ after solving the full system of equation.

System of equations in the reference configuration

Tensors $\boldsymbol{\tau}$, $\mathbf{e}$, $\mathbf{J}$ and vectors $\boldsymbol{\omega}$, $\mathbf{v}$ were considered as a basis for the CC. To write down the system of equation for the RC we need to use tensors and vectors as a basis for the RC.

For the RC we need to use "rotated" velocities vector

$$\mathbf{V} = \mathbf{P}^T\cdot\mathbf{v} \tag{15}$$

$$\boldsymbol{\Omega} = \mathbf{P}^T\cdot\boldsymbol{\omega} \tag{16}$$

The vector $\boldsymbol{\Omega}$ is used in rigid body dynamic (Zhilin 1996). There it was called the right angular velocity vector and is defined by the equation

$$\dot{\mathbf{P}} = \mathbf{P}\times\boldsymbol{\Omega} \tag{17}$$

For the CC tensor $\mathbf{J}$ was defined as follows $\mathbf{J} = \mathbf{P}\cdot\mathbf{J}_0\cdot\mathbf{P}^T$ (Zhilin 1996), where $\mathbf{J}_0$ is the known mass density of an inertia tensor for the RC. The stress state for the RC for the reduced Cosserat continuum is described by the stress tensor

$$\mathbf{T} = \frac{\rho_0}{\rho}\mathbf{F}^{-1}\cdot\boldsymbol{\tau}\cdot\mathbf{P}, \tag{18}$$

where ρ_0 is a density in RC, $\mathbf{F} = \overset{\circ}{\nabla} \mathbf{r}^T$.

The strain state for the is described by the strain tensor

$$\mathbf{E} = \mathbf{F}^T \cdot \mathbf{P} - \mathbf{I} \tag{19}$$

This strain tensor is identically equal to zero when body moves as rigid.

Define V_0 as a volume for the RC which changes in V for the CC. Volumes V and V_0 consist of the same particles. We use the linear momentum balance equation (1), the Gauss-Ostrogradskii theorem and the Nanson's formula

$$\mathbf{n} dS = \frac{\rho_0}{\rho} \mathbf{N} \cdot \mathbf{F}^{-1} dS_0, \tag{20}$$

where $\mathbf{n}$ is an outward unit normal to the surface S, $\mathbf{N}$ is an outward unit normal to the surface S_0. Because V_0 is arbitrary, we get

$$\rho_0 \dot{\mathbf{v}} = \overset{\circ}{\nabla} \cdot \mathbf{T} \cdot \mathbf{P}^T + \mathbf{R}^s \cdot \mathbf{T} \cdot \frac{\partial \mathbf{P}^T}{\partial x^s}. \tag{21}$$

We transform summands in the expression (21). For this, we introduce an additional tensor $\mathbf{K}$:

$$\overset{\circ}{\nabla} \mathbf{P}^T = -\mathbf{K} \times \mathbf{P}^T. \tag{22}$$

Hence $\frac{\partial P^T}{\partial x^s} = -\mathbf{K}_s \times \mathbf{P}^T$, because $\mathbf{K}$ satisfies a relation $\mathbf{K} = \mathbf{r}^s \mathbf{K}_s$.

$$\dot{\mathbf{v}} = (\mathbf{P} \cdot \mathbf{V})^{\dot{}} = (\mathbf{P} \times \mathbf{\Omega}) \cdot \mathbf{V} + \mathbf{P} \cdot \dot{\mathbf{V}} = (\dot{\mathbf{V}} + \mathbf{\Omega} \times \mathbf{V}) \cdot \mathbf{P}^T$$

$$\begin{aligned} \mathbf{R}^s \cdot \mathbf{T} \cdot \frac{\partial \mathbf{P}^T}{\partial x^s} &= -\mathbf{R}^s \cdot \mathbf{T} \cdot (\mathbf{K}_s \times \mathbf{P}^T) = \\ &= -(\mathbf{R}^s \cdot \mathbf{T} \times \mathbf{K}_s) \cdot \mathbf{P}^T = (\mathbf{K}^T \cdot \mathbf{T})_x \cdot \mathbf{P}^T \end{aligned} \tag{23}$$

where we use the identity $\mathbf{A}_x = -\mathbf{A}_x^T$ and $(\mathbf{A}^T \cdot \mathbf{K})_x = \mathbf{R}^k \cdot \mathbf{A} \times \mathbf{K}_k$, that is valid for any tensor $\mathbf{A}$.

Let's return to the equation (21) and multiply it by the tensor $\mathbf{P}$ on the right. As a result we obtain a local form of the linear momentum balance equation for the RC

$$\overset{\circ}{\nabla} \cdot \mathbf{T} + (\mathbf{K}^T \cdot \mathbf{T})_x = \rho_0 (\dot{\mathbf{V}} + \mathbf{\Omega} \times \mathbf{V}) \tag{24}$$

Now we consider the kinetic moment balancing equation (2). Let's write (2) for the RC using (16), (18), (20), the Gauss-Ostrogradskii theorem and arbitrariness of the volume V_0

$$\rho_0 (\mathbf{P} \cdot \mathbf{J}_0 \cdot \dot{\mathbf{\Omega}} + \mathbf{r} \times \mathbf{v}) = -\overset{\circ}{\nabla} \cdot (\mathbf{T} \cdot \mathbf{P}^T \times \mathbf{r}) \tag{25}$$

We transform summands in (25) $\mathbf{r} \times \mathbf{v} = \mathbf{r} \times (\dot{\mathbf{V}} + \mathbf{\Omega} \times \mathbf{V}) \cdot \mathbf{P}^T$. Also using the expression (23) we can get

$$\overset{\circ}{\nabla} \cdot (\mathbf{T} \cdot \mathbf{P}^T) \times \mathbf{r} = -\mathbf{r} \times (\overset{\circ}{\nabla} \cdot \mathbf{T} + (\mathbf{K}^T \cdot \mathbf{T})_x) \cdot \mathbf{P}^T.$$

$$\mathbf{R}^k \cdot \mathbf{T} \cdot \mathbf{P}^T \times \mathbf{r}_k = -\mathbf{P} \cdot ((\mathbf{E} + \mathbf{I})^T \cdot \mathbf{T})_x$$

$$(\mathbf{P} \cdot \mathbf{J}_0 \cdot \mathbf{\Omega})^{\dot{}} = \mathbf{P} \cdot (\mathbf{J}_0 \cdot \dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{J}_0 \cdot \mathbf{\Omega})$$

Then returning to the expression (25), using (24) we get a local form of the kinetic moment balance equation for the RC.

$$((\mathbf{E} + \mathbf{I})^T \cdot \mathbf{T})_x = \rho_0 (\mathbf{J}_0 \cdot \dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{J}_0 \cdot \mathbf{\Omega}) \tag{26}$$

Let's differentiate with respect to time the equation (19). For the RC basis does not depend on time. So, using the opportunity to reshuffle $\partial / \partial t$ and $\partial / \partial x$, we get $\dot{\mathbf{F}}^T = \overset{\circ}{\nabla} \mathbf{v}$. Considering expression (17) and $\dot{\mathbf{I}} = 0$ we get the following relation $\dot{\mathbf{E}} = \overset{\circ}{\nabla} \cdot \mathbf{P} + \mathbf{F}^T \cdot \mathbf{P} \times \mathbf{\Omega}$ (27)

We had transformed first summand from the equation (27) using (22) and (15):

$$\begin{aligned} \overset{\circ}{\nabla} \cdot \mathbf{P} &= \overset{\circ}{\nabla} (\mathbf{v} \cdot \mathbf{P}) - \overset{\circ}{\nabla} \mathbf{P}^T \cdot \mathbf{v} = \overset{\circ}{\nabla} \mathbf{V} + \mathbf{K} \times \mathbf{P}^T \cdot \mathbf{v} = \\ &= \overset{\circ}{\nabla} \mathbf{V} + \mathbf{K} \times \mathbf{V} \end{aligned} \tag{28}$$

We had transformed second summand from the equation (27) using (19) and finally we get

$$\dot{\mathbf{E}} = \overset{\circ}{\nabla} \mathbf{V} + \mathbf{K} \times \mathbf{V} + (\mathbf{E} + \mathbf{I}) \times \mathbf{\Omega}. \tag{29}$$

Let's derive an equation relating the $\mathbf{K}$ and $\mathbf{\Omega}$. Transposing both sides of the equation (17), we get

$$\begin{aligned} \overset{\circ}{\nabla} \dot{\mathbf{P}}^T &= -\overset{\circ}{\nabla} \mathbf{\Omega} \times \mathbf{P}^T - \mathbf{R}^s \mathbf{\Omega} \times \frac{\partial \mathbf{P}^T}{\partial x^s} = \\ &= -\overset{\circ}{\nabla} \mathbf{\Omega} \times \mathbf{P}^T + \mathbf{R}^s \mathbf{\Omega} \times (\mathbf{K}_s \times \mathbf{P}^T) \end{aligned}$$

To transform the second term we use the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{A}) = \mathbf{b} \times (\mathbf{a} \times \mathbf{A}) + (\mathbf{a} \times \mathbf{b}) \times \mathbf{A}$, which is valid for any $\mathbf{a}$, $\mathbf{b}$, $\mathbf{A}$ (Zhilin 1996). Then

$$\overset{\circ}{\nabla} \dot{\mathbf{P}}^T = -\overset{\circ}{\nabla} \mathbf{\Omega} \times \mathbf{P}^T + \mathbf{K} \times (\mathbf{\Omega} \times \mathbf{P}^T) - (\mathbf{K} \times \mathbf{\Omega}) \times \mathbf{P}^T \tag{30}$$

Let's differentiate with respect to time (22), so we get

$$(\overset{\circ}{\nabla} \mathbf{P}^T)^{\dot{}} = -\dot{\mathbf{K}} \times \mathbf{P}^T + \mathbf{K} \times (\mathbf{\Omega} \times \mathbf{P}^T) \tag{31}$$

For the RC basis does not depend on time, which leads $(\overset{\circ}{\nabla} \mathbf{P}^T)^{\dot{}} = \overset{\circ}{\nabla} (\mathbf{P}^T)^{\dot{}}$. We equate expressions (30) and (31). $\dot{\mathbf{K}} \times \mathbf{P}^T = (\overset{\circ}{\nabla} \mathbf{\Omega} + \mathbf{K} \times \mathbf{\Omega}) \times \mathbf{P}^T$. Hence

$$\dot{\mathbf{K}} = \overset{\circ}{\nabla} \mathbf{\Omega} + \mathbf{K} \times \mathbf{\Omega} \tag{32}$$

Equations (29) and (32) are compatibility equations in the RC.

We need an additional tensor $\mathbf{K}$ to be able to write the motion equations (24), (26) and the compatibility equations (29), (32) for the RC.

Lalin (2007) showed that tensors $\mathbf{T}$ and $\mathbf{E}$ satisfy the equation $\rho_0 \dot{\Pi} = \mathbf{T}^T \cdot\cdot \dot{\mathbf{E}}$ and that proves their energy coupling. Now we can write the constitutive equation for the RC as

$$\rho_0 \frac{\partial \Pi}{\partial \mathbf{E}} = \mathbf{T}. \tag{33}$$

Unknown number increases to 33 for the RC: 9 stresses $\mathbf{T}$, 9 strains $\mathbf{E}$, 6 velocities $\mathbf{V}$, $\mathbf{\Omega}$ and 9 components of the additional tensor $\mathbf{K}$. Corresponding equation are:

6 motion equations (24), (26), 18 compatibility equations (29), (32), 9 constitutive equations (33). In total there are 33 equations.

Conclusions

We considered the nonlinear reduced Cosserat continuum. The following results have been presented:

1. Energy coupling stress and strain tensors were defined and used for the CC and the RC.
2. Compatibility equations of strains and velocities were obtained.
3. Two systems of equations for the RC and the CC were obtained.

Why is the reduced Cosserat continuum important for practical use? The reduced Cosserat model can be applied for description of granular media. The theory of granular media is applicable to soil mechanics: for soils consisting of large particles and clay soils (Badanin et al. 2012).

The following tasks for further research have been set:

1. To obtain a variational formulation for the presented systems of equations.
2. To study the conditions of the uniqueness of the solutions of nonlinear dynamic problems.

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